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Comparison principles for free-surface flows with gravity

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This article considers certain two-dimensional, irrotational, steady flows in fluid regions of finite depth and infinite horizontal extent. Geometrical information about these flows and their singularities is obtained, using a variant of a classical comparison principle. The results are applied to three types of problems: (i) supercritical solitary waves carrying planing surfaces or surfboards, (ii) supercritical flows past ship hulls and (iii) supercritical interfacial solitary waves in systems consisting of two immiscible fluids.

1. Introduction

A classical mathematical technique in the study of steady irrotational fluid motion is the use of comparison flows (e.g. Gilbarg 1960). In this article we report on a variant of the method for two-dimensional flows, with several new applications to free-surface flows with gravity. By choice of the proper comparison flows, geometrical information about the fluid region is obtained. The main results are for supercritical flows in fluid regions of finite depth and infinite horizontal extent. It is in this regime that solitary waves are observed. The applications presented here are to three types of problems: (i) free-surface flows past ship hulls in channels of finite depth, (ii) surfboards or other planing surfaces in solitary-wave-like flows, and (iii) waves in an interface between immiscible fluids in a system of two fluids with differing densities. Comparison flows are most easily obtained from considering the fluid equations and boundary conditions in complex velocity potential coordinates. Later in the introduction our comparison principle will be stated. However, as the details of these three problems differ, the particular descriptions of the applications of the method are left to the sections below.

The use of comparison flows in the study of free boundary problems is well known. A common reference is Gilbarg (1960), with applications by many authors including Lewy (1952), Serrin (1954), Keady & Pritchard (1974) and Caffarelli (1986). In the theory of steady flows with gravity, comparison principles were applied to the classical solitary wave (Craig & Sternberg 1988), with the conclusion that all solitary waves in the free surface of a fluid in a channel of fixed depth and infinite horizontal extent are positively elevated above the asymptotic fluid level, symmetric about a unique crest, and monotone on either side of the crest. This article presents certain extensions of the method to the three problems mentioned above. In all cases, our approach applies to the situation where the solitary wave can be represented globally as the graph of a function.

Two of the results are worthy of particular note. The first is that, for certain profile geometries of ship hull, there are no supercritical steady flows which do not have splash, spray or other singularities in the free surface. Since this has bearing on the resistance of these objects to motion, there may be design consequences for such profiles when supercritical velocities are expected.

The second result concerns solitary waves in systems of two immiscible fluids of different densities ρ_1 and ρ_2. It is well-known (e.g. Benjamin 1966) that there is an analogue of the critical Froude number, dependent on the channel depth, the asymptotic relative depths h_1, h_2 of the two fluids at infinity, and the ratio of their densities, ρ_1/ρ_2. Furthermore, in small-amplitude theory (Benjamin) it is known that supercritical solitary waves are elevated (resp. depressed) with respect to the asymptotic level, depending upon the inequality $\rho_1/h_1^2 < \rho_2/h_2^2$ (resp. $\rho_1/h_1^2 > \rho_2/h_2^2$). We show that this result holds for supercritical waves of any amplitude. Additionally, in the case when the ratios are equal, we show that there can be no supercritical solitary waves. Thus whenever $\rho_1/h_1^2 = \rho_2/h_2^2$, solutions which have solitary-wave-like profiles must either not be steady flows, or travel at subcritical speeds, at which they are unstable to radiating a wake at infinity.

We have obtained more detailed mathematical results for the three problems mentioned above; these will appear in a subsequent publication (Craig & Sternberg 1991). The results relate primarily to the monotonicity and symmetry of free-surface profiles, and their asymptotic behaviour at infinity.

The variant of the classical comparison principle that we use in this article is based on the complex velocity potential. Any incompressible flow with no vorticity has a velocity potential that is harmonic,

$$\Delta\Phi = 0. \tag{1}$$

In two-dimensional flows this is joined with its harmonic conjugate function Ψ to make the complex velocity potential. The inverse of this analytic function is again analytic, a conformal mapping of a certain domain S in the complex velocity potential plane to the physical fluid domain,

$$Z(\Phi+\mathrm{i}\Psi) = (X+\mathrm{i}Y)(\Phi+\mathrm{i}\Psi). \tag{2}$$

Of course, both X and Y are harmonic functions of (Φ, Ψ). Since the complex velocity potential is defined only up to a constant, the corresponding domain determines the same flow under any translation. Consider two steady flows, and suppose that, possibly after an initial translation, the corresponding domains coincide, $S_1 = S_2$. The comparison principle concerns the values of (X_1, Y_1) and (X_2, Y_2) in S_1. Since non-constant harmonic functions cannot have interior maxima or minima (e.g. Protter & Weinberger 1967), the first statement is that if $Y_1 \leqslant Y_2$ throughout S_1, then

$$Y_1 < Y_2 \tag{3}$$

everywhere interior to S_1, or else the flows coincide, and there is a real constant R such that $Z_1 = Z_2 + R$. Actually, it suffices to demand that $Y_1 \leqslant Y_2$ only at boundary points of S_1 in order to draw this first conclusion. A similar statement can be made for X.

The second comparison result concerns the boundary behaviour of the conformal mapping. Assume that

$$Y_1 \leqslant Y_2 \tag{4}$$

at all boundary points of S_1, and that at a particular boundary point (Φ_0, Ψ_0) equality holds:

$$Y_1(\Phi_0, \Psi_0) = Y_2(\Phi_0, \Psi_0). \tag{5}$$

Then the tangential derivatives coincide,

$$\partial_{\mathrm{T}}(Y_2 - Y_1)(\Phi_0, \Psi_0) = 0, \tag{6}$$

and additionally, either the exterior normal derivatives satisfy

$$\partial_{\mathrm{N}}(Y_2 - Y_1)(\Phi_0, \Psi_0) < 0, \tag{7}$$

or else again the flows coincide, $Z_1 = Z_2 + R$. This is known as the Hopf boundary point lemma for harmonic functions (e.g. Protter & Weinberger 1967). In the applications below, the complex velocity potential domain is a strip

$$\{\Phi + \mathrm{i}\Psi : -\infty < \Phi < \infty, -ch \leqslant \Psi \leqslant 0\},$$

where c is the wave speed and h measures asymptotic depth of the fluid. The free surface corresponds to the top $\Psi = 0$, and the tangential and normal derivatives of Y are respectively $\partial_{\Phi} Y$ and $\partial_{\Psi} Y$.

This comparison result is somewhat stronger than Gilbarg's result (1960, Theorem 1, section 27) in the two-dimensional case. To compare them, note that if u and v are the horizontal and vertical components of velocity, then one readily calculates that

$$\partial_{\mathrm{T}} Y = \frac{v}{u^2 + v^2}, \quad \partial_{\mathrm{N}} Y = \frac{u}{u^2 + v^2}.$$

Restating (6) and (7) again under the hypotheses (4) and (5), we find that at the point (Φ_0, Ψ_0) we have

$$\partial_{\mathrm{T}} Y_1 = \frac{v_1}{u_1^2 + v_1^2} = \frac{v_2}{u_2^2 + v_2^2} = \partial_{\mathrm{T}} Y_2, \tag{8}$$

and

$$\partial_{\mathrm{N}} Y_1 = \frac{u_1}{u_1^2 + v_1^2} > \frac{u_2}{u_2^2 + v_2^2} = \partial_{\mathrm{N}} Y_2, \tag{9}$$

which implies that $u_1^2 + v_1^2 < u_2^2 + v_2^2$, or else the flows are equivalent. This is Gilbarg's comparison of flow speeds, for if the physical boundaries of the two flows coincide at $(\Phi_0, \Psi_0)_{\mathrm{S}}$, the maximum modulus principle implies that the second fluid region contains the first. Using (6) again, we find that $|v_1| < |v_2|$, a new statement comparing vertical velocities of the two fluids. It is conditions (6) and (7) that we use in the subsequent discussion.

2. Free-surface problems with objects in the surface

We consider first an impermeable solid object such as a ship hull or planing surface of given shape which is taken to lie upon the top surface of a steady two-dimensional flow. This leads to a boundary-value problem of mixed type. We assume that the flow has no spray or splash and that the top surface is representable as the graph of a function, so that the fluid region is described as $\{(x, y): -\infty < x < +\infty, -h < y < \Gamma(x)\}$. For related numerical studies of such problems we refer to Keller & Vanden-Broeck (1989) and Vanden-Broeck (1987). The surface of the fluid region consists of two subregions: one is in contact with the solid object, while in the other region the surface is free and so must satisfy the Bernoulli condition stating the continuity of pressure. The upper fluid boundary $(x, \Gamma(x))$ and the velocity potential $\Phi(x, y)$ which describe this steady flow satisfy:

$$\left.\begin{array}{l}\partial_n \Phi = 0 \text{ for all } (x, \Gamma(x)) \text{ on the upper fluid boundary,} \\ \Gamma(x) \text{ is prescribed by the solid object on the fixed portion} \\ \qquad\qquad \text{of the upper fluid boundary,} \\ \tfrac{1}{2}(\nabla\Phi)^2 + g\Gamma = \tfrac{1}{2}c^2 \text{ for } (x, \Gamma(x)) \text{ on the free boundary of the} \\ \qquad\qquad \text{upper fluid boundary,} \\ \Delta\Phi = 0 \quad \text{for} \quad -\infty < x < +\infty,\ -h < y < \Gamma(x), \\ \partial_n \Phi = 0 \quad \text{for} \quad y = -h. \end{array}\right\} \tag{10}$$

We address the problem of flows which approach uniform flow at infinity, so that we take as asymptotic conditions

$$\left.\begin{array}{r}\Phi(x,y) \rightarrow -cx, \\ \nabla\Phi(x,y) \rightarrow (-c, 0)\end{array}\right\} \tag{11}$$

as $x \rightarrow \pm\infty$. This implies by (10) that the free-surface elevation $\Gamma(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. Furthermore, we may apply the Hopf boundary point lemma (cf. (7)) to conclude that on the top surface,

$$\partial_y \Psi(x, \Gamma(x)) > 0.$$

Similarly,

$$\partial_y \Psi(x, -h) > 0.$$

Since, the asymptotic conditions imply that $\partial_y \Psi > 0$ as $|x| \rightarrow \pm\infty$, the first comparison principle, (cf. (3)), yields the inequality

$$\partial_y \Psi > 0 \tag{12}$$

throughout the fluid region. In particular, note that (12) precludes the possibility that a streamline develops a vertical slope.

The boundary-value problem (10), (11) is equivalent to a nonlinear problem for the conformal mapping of a fixed strip into the fluid region. This is the complex velocity potential plane; the fluid equations are transformed so that the independent variables are the velocity potential Φ and the stream function Ψ.

We find it more convenient here to introduce variables $(c\xi, c\eta) = (\Phi, \Psi)$ in the strip

$$S = \{(\xi,\eta): -\infty < \xi < +\infty,\ -h < \eta < 0\}.$$

The conformal map is $Z(\xi,\eta) = (X+\mathrm{i}Y)(\xi,\eta)$, taking the strip S to the fluid domain. By the asymptotic conditions (11), we have $(X, Y) \rightarrow (\xi, \eta)$ as $|\xi| \rightarrow \infty$. Let

$$X(\xi,\eta) = \xi + x(\xi,\eta), \quad Y(\xi,\eta) = \eta + y(\xi,\eta),$$

where $(x+\mathrm{i}y)(\xi,\eta)$ is the perturbation of the conformal map from the one for a uniform flow. Under this transformation, the top boundary $\partial = \{(\xi, 0), -\infty < \xi < \infty\}$ of S is transformed into the upper fluid boundary of the fluid region, and thus ∂ is divided into two sets. One is the set which is transformed by Z onto the free surface; we denote it by ∂_1. The remaining region ∂_2 is the set which is transformed to the boundary of the contact between the fluid and the object in the surface. The

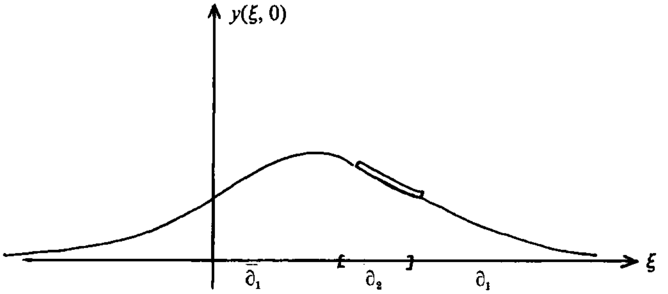

FIGURE 1. Object in top surface above mean level.

set ∂_1 is taken to be closed, and ∂_2 to be bounded. In these coordinates, the problem is most conveniently posed for the function $y(\xi, \eta)$ in the region S:

$$\left.\begin{aligned} \frac{1}{(\partial_\xi y)^2+(1+\partial_\eta y)^2}+\frac{2g}{c^2}y &= 1 \quad \text{for} \quad \eta = 0, \quad \xi \in \partial_1, \quad \text{the free surface,} \\ y(\xi,0) &= \Gamma(X(\xi,0)) \quad \text{for} \quad \eta = 0, \quad \xi \in \partial_2, \quad \text{the fixed boundary,} \\ \Delta y &= 0 \quad \text{for} \quad (\xi,\eta) \in S, \\ y(\xi, -h) &= 0, \quad \text{bottom boundary conditions.} \end{aligned}\right\} \tag{13}$$

The asymptotic condition becomes

$$(y, \partial_\xi y, \partial_\eta y) \to 0 \quad \text{as} \quad \xi \to \pm\infty. \tag{14}$$

The transformation to the complex velocity potential plane, and the equivalence of (10), (11) to (13), (14) can be established if one assumes that the solution to (10), (11) has continuous derivatives. We shall make this assumption, which precludes the possibility that streamlines terminate or reverse direction. If no physical evacuation of the channel occurs, that is, if $Y(\xi,0) > -h$, then from (12) one immediately obtains the bound

$$\partial_\eta y(\xi,\eta) > -1 \quad \text{for all} \quad (\xi,\eta) \in S, \tag{15}$$

stating that streamlines never develop vertical tangents.

We first note an upper bound on the height above the asymptotic fluid-level attainable by a point on the free surface. Clearly

$$\frac{1}{(\partial_\xi y)^2+(1+\partial_\eta y)^2} > 0,$$

thus for any $\xi_0 \in \partial_1$, the boundary conditions in (13) imply that

$$y(\xi_0, 0) < \frac{c^2}{2g}. \tag{16}$$

If equality occurs in (16), $(\partial_\xi Y)^2+(\partial_\eta Y)^2$ is infinite, and the surface must possess a singularity, the crest of the Stokes wave of extremal form. This estimate is well-known in other settings (Staar 1947; Keady & Pritchard 1974) but is particularly simple to see in these variables.

2.1. *Objects above mean level*

Examine the situation where the object in the top surface lies entirely above mean level, so that
$$y(\xi,0) > 0 \quad \text{for} \quad \xi \in \partial_2 \tag{17}$$
(see figure 1). This would be relevant, for example, in the case of a surfboard or some other planing object riding on the top surface of the fluid. We shall derive a bound for the highest crest and lowest trough if it occurs on the free surface, and in so doing, we will assume that the wave is propagating at a supercritical speed, $c^2 \geqslant gh$. For subcritical speeds, there is some information about the amplitude of solutions through the comparison principle, however, it is less conclusive and it is not pursued in this paper. Suppose first that $y(\xi_0, 0) > 0$ is the highest crest, being attained at a point $\xi_0 \in \partial_1$, on the free boundary. Compare this solution to the uniform flow given by the linear function $l(\xi,\eta) = [y(\xi_0,0)/h](\eta+h)$. Clearly $l(\xi,0) \geqslant y(\xi,0)$, $l(\xi,-h) = y(\xi,-h) = 0$ and $l(\xi,\eta) > y(\xi,\eta) \to 0$ as $\xi \to \pm\infty$. Use of the comparison principles in §1 implies first that $l(\xi,\eta) > y(\xi,\eta)$ for all $(\xi,\eta) \in S$, as well as a comparison of the normal derivatives of l and y at $(\xi_0, 0)$:
$$\left.\begin{aligned} &\partial_\eta y(\xi_0,0) > \partial_\eta l(\xi_0,0) = \frac{y(\xi_0,0)}{\eta}, \\ &\partial_\xi y(\xi_0,0) = 0. \end{aligned}\right\} \tag{18}$$
Since $\xi_0 \in \partial_1$, this strict inequality is used in the Bernoulli condition
$$1 = \frac{1}{(1+\partial_\eta y)^2 + (\partial_\xi y)^2} + \frac{2g}{c^2} y < \frac{1}{(1+m)^2} + \frac{2ghm}{c^2},$$
where $y(\xi_0,0)/h \equiv m > 0$ is used. This can be rewritten as
$$m^2 + (2-\tfrac{1}{2}F^2)m + (1-F^2) > 0 \tag{19}$$
for $F^2 = c^2/gh$, the square of the Froude number. In the supercritical regime one has $F > 1$, and the rightmost root of the quadratic (19), which we denote by $m^+(c)$, is positive, while the left most root, $m^-(c)$, is negative. Since $m > 0$, (19) implies that $m > m^+(c)$, and we obtain a lower bound on the maximum amplitude of a solution if it is attained on the free portion of the surface, ∂_1. This result can also be expressed as $F^2 < 2(m+1)^2/(m+2)$, which has appeared in Staar (1947) in the context of the solitary wave.

In this elevated setting (17), consider now the possibility that a lowest trough of a solution of (13) is attained somewhere on the free surface, and lies below the asymptotic level of the top fluid surface. That is, suppose there exists $\xi_0 \in \partial_1$ such that $y(\xi_0, 0) < 0$ is the lowest trough. Again using the uniform flow
$$l(\xi,\eta) = [y(\xi_0,0)/h](\eta+h),$$
one can apply the comparison principle to obtain:
$$\left.\begin{aligned} &\partial_\eta y(\xi_0,0) < \partial_\eta l(\xi_0,0) = \frac{y(\xi_0,0)}{h} < 0, \\ &\partial_\xi y(\xi_0,0) = 0. \end{aligned}\right\} \tag{20}$$
Substitution into the Bernoulli condition then implies
$$1 - \frac{2g}{c^2} y(\xi_0,0) = \frac{1}{(1+\partial_\eta y)^2 + (\partial_\xi y)^2} > \frac{1}{(1+m)^2},$$

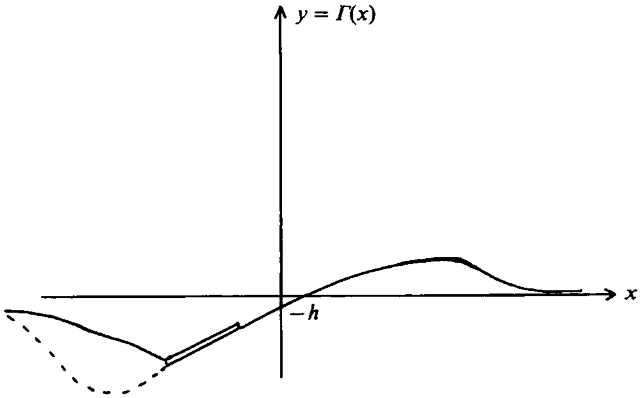

FIGURE 2. Singularity at a minimum (solid line). The dashed line is a smooth flow which is disallowed since the minimum occurs on the free surface.

where now $y(\xi_0, 0)/h \equiv m < 0$. Here we have used (15), and we conclude that the minimum value m satisfies the inequality

$$1 - \frac{2gh}{c^2} m > \frac{1}{(1+m)^2}, \tag{21}$$

which for $m < 0$ is equivalent to (19). We learn that $m > m^+(c)$. However, recall that for supercritical c, $m^+(c) \geqslant 0$. This is a contradiction, the resolution of which precludes the occurrence of a trough with a minimum below mean level attained on the free surface for supercritical flows. In other words, for $F \geqslant 1$, if the object lies above the asymptotic level of the fluid, then the entire top surface is elevated as well. In the absence of any object, we recover the fact that supercritical solitary waves are positive, established in Craig & Sternberg (1988).

2.2. *Objects below mean level*

Turning to the setting relevant to a ship hull, we consider the case where some portion of the object lies depressed below the asymptotic fluid level. Assume that the velocity is supercritical, and that the flow has no spray or splash. Thus there is no wake at infinity, and the fluid surface $\{y = \Gamma(x)\}$ tends to horizontal as $x \to \pm\infty$. Using the arguments of the preceding section, in the hypothetical situation in which the lowest point on the fluid surface is attained on the free boundary, (21) leads to a contradiction. The resolution of this is that either the minimum point is attained on the solid object, or the flow has a singularity in the form of spray or splash. Refining the argument slightly one can show that if the free surface makes contact with the object with continuous tangent, then the minimum of the fluid surface cannot occur at these points of contact.

Thus for smooth supercritical flows, any minimum attained by the fluid surface must occur only in the interior of the object, not on the free surface or at the endpoints of the object. This is a restriction on the geometries of the solid object which allow smooth free-surface flows at supercritical velocities. In particular, if the specified profile of the object $\{(x, \Gamma(x)) : a \leqslant x \leqslant b\}$ has a minimum point below mean level at either endpoint of the region in contact with the fluid, then the flow cannot have a continuous derivative. The remaining possibilities are that the flow has splash, spray or a boundary singularity at the leading or trailing edge. Such

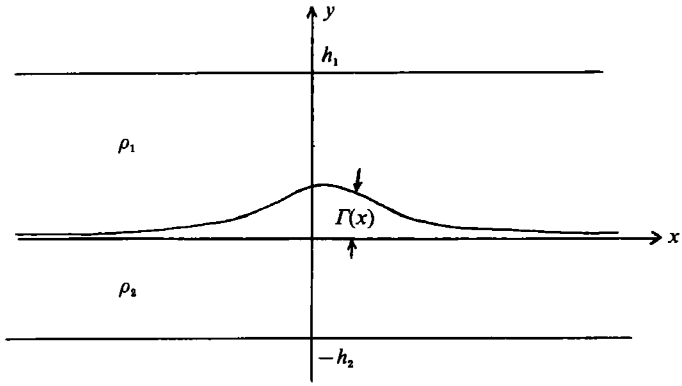

FIGURE 3. Interfacial wave between two immiscible fluids.

singularities in a real flow certainly have an effect on the hydrodynamical drag and the forces exerted on the solid object. In the case of a hull, geometrical considerations in its design could reduce or eliminate these singularities. (See figure 2).

3. Waves in an interface between two immiscible fluids

Similar comparison techniques also apply to the problem of waves in an interface between two horizontally infinite bodies of fluid, again under the influence of gravity. We consider two-dimensional steady flow, asymptotic to uniform flow as $x \to \pm\infty$, with the same speed c for each fluid. Equivalently, this describes a wave propagating without change of form in this double layer of fluid which is taken at rest at $x \to \pm\infty$.

We choose to retain four physical parameters in this problem: the asymptotic depths of the fluid layers, h_1 and h_2, as $x \to \pm\infty$, and the fluid densities ρ_1 and ρ_2. The density ρ_1 will refer to the upper fluid density, which will be taken less than ρ_2. (See figure 3).

Denoting the free interface by $\Gamma(x)$ and the velocity potentials for the upper and lower fluid bodies by $\Phi_1(x,y)$, $\Phi_2(x,y)$, one poses the problem:

$$\left.\begin{aligned} \partial_y \Phi_1 &= 0 \quad \text{for} \quad y = h_1, \\ \partial_y \Phi_2 &= 0 \quad \text{for} \quad y = -h_2, \\ \Delta\Phi_1 &= 0, \quad -\infty < x < +\infty, \quad \Gamma(x) < y < h_1, \\ \Delta\Phi_2 &= 0, \quad -\infty < x < +\infty, \quad -h_2 < y < \Gamma(x), \\ \rho_1(\tfrac{1}{2}(\nabla\Phi_1)^2 + \Gamma(x) - \tfrac{1}{2}c^2) &= \rho_2(\tfrac{1}{2}(\nabla\Phi_2)^2 + \Gamma(x) - \tfrac{1}{2}c^2) \quad \text{for} \quad y = \Gamma(x), \end{aligned}\right\} \tag{22}$$

where the last formula is the Bernoulli condition maintaining continuity of pressure across the interface. As before, we consider solutions that approach uniform flow at infinity; that is we impose the asymptotic conditions

$$\left.\begin{aligned} &\Phi_j(x,y) \to -cx, \\ &\nabla\Phi_j(x,y) \to (-c, 0), \quad j = 1,2, \end{aligned}\right\} \text{ as } x \to \pm\infty. \tag{23}$$

We wish to emphasize that again we assume that the interface can be described as the graph of a function, $\Gamma(x)$, in order to be able to apply our comparison principles. There is numerical evidence to suggest that some interfacial solitary waves can develop vertical slopes or 'overhang' (e.g. Meiron & Saffman 1983; Grimshaw &

Pullin 1986; Turner & Vanden-Broeck 1988). However, in the present setting, we are unable to treat these waves with our techniques. As in the previous section (cf. (12)) this assumption on Γ first allows one to apply the comparison principles to conclude that streamlines do not develop vertical tangents,

$$\partial_y \Psi_j > 0, \quad j = 1,2, \tag{24}$$

throughout the two fluid regions.

Our goal here is to apply the comparison arguments to obtain information on the interfacial wave; in particular as to its elevation or depression, and its amplitude. To this end we find it most convenient to introduce coordinates analogous to the complex velocity potential transformation; that is, one considers the equivalent problem of finding conformal mappings from fixed domains onto the regions occupied by the two fluids. Let (Φ_j, Ψ_j), $j = 1,2$, be the velocity potential and stream function for the two fluid regions, with $\Psi_1(x,\Gamma(x)) = \Psi_2(x,\Gamma(x)) = 0$ for normalization. Define two fixed domains

$$S_1 = \{(\xi,\eta): -\infty < \xi < +\infty, 0 < \eta < h_1\},$$
$$S_2 = \{(\xi,\eta): -\infty < \xi < +\infty, -h_2 < \eta < 0\},$$

and set $(c\xi, c\eta) = (\Phi_1, \Psi_1)$ in S_1, $(c\xi, c\eta) = (\Phi_2, \Psi_2)$ in S_2. The problem (22), (23) is equivalent to a system of differential equations for the conformal maps $Z_j = (X_j + \mathrm{i}Y_j)(\xi,\eta)$, $j = 1,2$, from the fixed domains S_1, S_2 into the regions occupied by the two fluids. Writing $Z_j = (\xi + x_j(\xi,\eta), \eta + y_j(\xi,\eta))$, one can restate the problem in terms of $y_j(\xi,\eta)$:

$$\left.\begin{aligned} \Delta y_1 &= 0 \quad \text{in} \quad S_1, \\ y_1(\xi,h_1) &= 0 \quad \text{for} \quad -\infty < \xi < +\infty, \text{ top boundary conditions}, \\ \Delta y_2 &= 0 \quad \text{in} \quad S_2, \\ y_2(\xi,-h_2) &= 0 \quad \text{for} \quad -\infty < \xi < +\infty, \text{ bottom boundary conditions}, \\ (y_j(\xi,\eta), &\partial_\xi y_j(\xi,\eta)\, \partial_\eta y_j(\xi,\eta)) \to 0 \quad \text{as} \quad \xi \to \pm\infty. \end{aligned}\right\} \tag{25}$$

It is also necessary to match y_j across the interface. Hence, we define $l(\xi)$ through the relation $X_2(l(\xi),0) = X_1(\xi,0)$ so that

$$y_1(\xi,0) = \Gamma(X_1(\xi,0)) = \Gamma(X_2(l(\xi),0)) = y_2(l(\xi),0), \tag{26}$$

which matches the vertical displacements of the interface. Finally, the Bernoulli condition on the interface becomes

$$\rho_1\left(\frac{1}{(\partial_\xi y_1)^2 + (1+\partial_\eta y_1)^2} + \frac{2g}{c^2}y_1 - 1\right)\bigg|_\xi = \rho_2\left(\frac{1}{(\partial_\xi y_2)^2 + (1+\partial_\eta y_2)^2} + \frac{2g}{c^2}y_2 - 1\right)\bigg|_{l(\xi)}. \tag{27}$$

This double complex velocity potential transformation is equivalent to the problem (22), (23). Furthermore, if neither fluid layer is breached,

$$-h_2 < \Gamma(x) < h_1, \tag{28}$$

then for smooth flows we use (24) to derive the estimate from below:

$$\partial_\eta y_j(\xi,\eta) > -1, \quad j = 1,2. \tag{29}$$

Using this formulation, we now obtain detailed information on the sign and amplitude of these interfacial waves in terms of the parameter values ρ_1, ρ_2, h_1 and

h_2. The strongest results are obtained for supercritical velocities: $c^2 > c_0^2$, where the critical velocity is defined (e.g. Benjamin 1966) to be

$$c_0^2 = \frac{(\rho_2-\rho_1)g}{(\rho_1/h_1+\rho_2/h_2)}. \tag{30}$$

We note that it is only in this regime that the existence of interfacial solitary waves has been established (Amick & Turner 1986; Bona & Sachs 1989). The significance of the critical velocity comes from the fact that for $c^2 > c_0^2$, steady flows which approach uniform flow at infinity cannot support small-amplitude periodic ripples at infinity. Furthermore, the comparison principle employed below yields weaker information in the subcritical case, $c < c_0$.

With an eye towards establishing a dichotomy, based on the physical parameter values, between waves of elevation and waves of depression which solve (22), (23) we suppose that $y_1(\xi, 0)$ attains a non-negative maximum at the value ξ_0. By translation, we may assume that $\xi_0 = 0$ and $l(\xi_0) = 0$ so that

$$M \equiv \max_{\xi} y_1(\xi,0) = y_1(0,0) = y_2(0,0) \geqslant 0.$$

Then $\partial_\xi y_j(0,0) = 0$ for $j = 1,2$, and we may compare separately y_1 and y_2 to the corresponding uniform flows having the same asymptotic velocities. Use of the comparison principle of §1 yields

$$y_1(\xi,\eta) < -M/h_1(\eta-h_1) \quad \text{in} \quad S_1,$$
$$y_2(\xi,\eta) < M/h_2(\eta+h_2) \quad \text{in} \quad S_2.$$

Then (7) gives the inequalities

$$\partial_\eta y_1 < -M/h_1, \quad \partial_\eta y_2 > M/h_2.$$

Making the physical assumption $M < h_1$, we may use these conditions, along with (29), in the Bournoulli condition (27) to obtain an inequality for M:

$$\rho_1\left(\frac{h_1^2}{(h_1-M)^2}+\frac{2gM}{c^2}-1\right) < \rho_2\left(\frac{h_2^2}{(h_2+M)^2}+\frac{2gM}{c^2}-1\right). \tag{31}$$

This can be rephrased in the more convenient form

$$a(M) < b(M,c) \quad \text{for} \quad 0 \leqslant M, \tag{32}$$

where one defines
$$a(s) = \frac{\rho_1 h_1^2}{(h_1-s)^2} - \frac{\rho_2 h_2^2}{(h_2+s)^2}, \tag{33}$$

and
$$b(s,c) = (\rho_2-\rho_1)(2g/c^2)(s-1).$$

In an entirely analogous manner, if one assumes the presence of a non-positive minimum m for the interfacial wave, then a comparison to uniform flow yields the condition

$$a(m) > b(m,c) \quad \text{for} \quad m \leqslant 0. \tag{34}$$

We can now draw numerous conclusions valid for the supercritical regime, $c \geqslant c_0$, under hypothesis (28), based on the following readily verified facts about the functions a and b:

(i) $a(0) = b(0,c)$ for all c;

(ii) $a'(s) > 0$ for $-h_2 < s < h_1$;

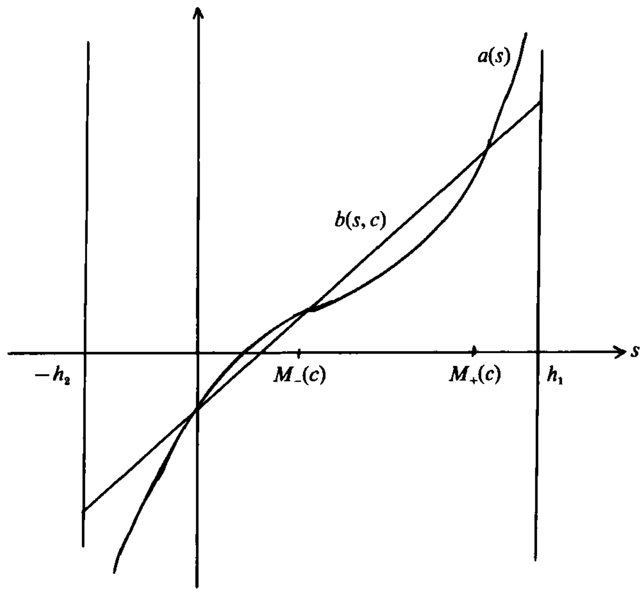

FIGURE 4. Supercritical regime allowing only elevation waves.

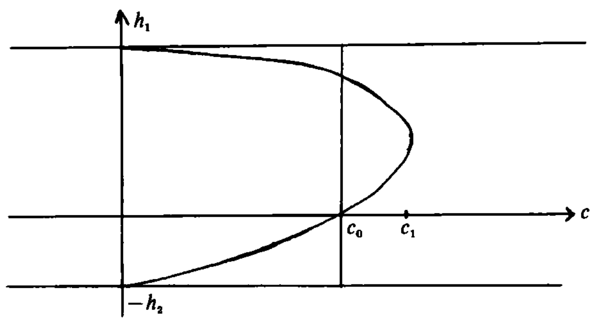

FIGURE 5. Upper bound c_1 on the fastest wave of elevation. The upper branch of the curve is the graph of $M_+(c)$ and the lower branch is the graph of $M_-(c)$. The region bounded by this curve and the vertical line $c = c_0$ contains the set of possible points (c, M), where c is wave speed and M is amplitude of an interfacial solitary wave of elevation.

(iii) $a'(0) > \partial_s b(0,c)$ for $c > c_0$, with $a'(0) = \partial_s b(0, c_0)$;
(iv) $a(s)$ has only one inflexion point in the interval $-h_2 < s < h_1$;
(v) $\lim_{s \to h_1} a(s) = \infty$, $\lim_{s \to -h_2} a(s) = -\infty$.

3.1. *Conclusions for interfacial waves*

(i) *Elevation waves.* Suppose that

$$a''(0) = 6\left(\frac{\rho_1}{h_1^2} - \frac{\rho_2}{h_2^2}\right) < 0. \tag{35}$$

Then the graph of $a(s)$ lies below that of $b(s, c)$ for all $s < 0$, precluding the possibility of a negative minimum m which would satisfy (34) (see figure 4). Thus $\Gamma(x) > 0$ and the wave is one of elevation. From (32) we learn that the amplitude M of this elevation wave satisfies the bounds

$$M_-(c) < M < M_+(c), \tag{36}$$

where $M_-(c)$ and $M_+(c)$ are the two positive roots to the equation $a(s) = b(s, c)$. Since

$$\mathrm{d}/\mathrm{d}c M_+(c) < 0, \tag{37}$$

it follows that any supercritical wave of elevation has amplitude bounded from above by $M_+(c_0)$. Furthermore, there is an upper bound c_1 on the velocity of the fastest wave of elevation, given implicitly by the requirement that at the upper bound c_1, the graphs of $a(s)$ and $b(s, c_1)$ intersect tangentially at a positive value of s (see figure 5). The bounds $M_\pm(c)$ and c_1 are algebraic functions in the parameters $(\rho_1, \rho_2, h_1, h_2)$, as one sees from (33).

(ii) *Depression waves.* Suppose now that

$$\rho_1/h_1^2 - \rho_2/h_2^2 > 0. \tag{38}$$

Then the graph of $a(s)$ lies above that of $b(s, c)$ for all $s > 0$, precluding the existence of a positive maximum satisfying (32). Hence $\Gamma(x) < 0$ and the solution must be a wave of depression. Applying (34) in this setting, we obtain bounds on the (negative) amplitude m of such a depression wave:

$$m_-(c) < m < m_+(c),$$

where $m_-(c)$ and $m_+(c)$ are the two negative roots to the equation $a(s) = b(s, c)$. Since

$$\mathrm{d}/\mathrm{d}c\, m_-(c) > 0, \tag{39}$$

it follows that any supercritical wave of depression falls no lower than $m_-(c_0)$. As before, one also derives a bound, c_2, for the fastest possible speed of a wave of depression.

These two global conclusions extend the results of Benjamin from the small-amplitude regime to the case of solitary waves of any amplitude.

(iii) *The case of equality.* In the remaining case

$$\rho_1/h_1^2 = \rho_2/h_2^2, \tag{40}$$

we find that $a(s) > b(s, c)$ for $s > 0$, while $a(s) < b(s, c)$ for $s < 0$, precluding by (32) and (34) any deviation of the surface toward elevation or depression above the mean level. Thus the trivial solution $\Gamma(x) = 0$ is the only possibility. This completes the picture of permissible amplitudes and wave speeds for solitary waves in interfaces.

In the light of (37) and (39), this analysis suggests that the presence of upper and lower boundaries forces the larger amplitude waves to slow down. In the third case, (40), supercritical speeds are disallowed. Finally, we note that the clear distinction between waves of elevation and depression given by conditions (35) and (38) provides a crucial first step in the proof of *a priori* symmetry of the interfacial waves profiles (Craig & Sternberg 1991).

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